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No-ghost theorem for the fourth-order derivative Pais-Uhlenbeck oscillator model

Carl M. Bender[1]* and Philip D. Mannheim[2]

[1] *Center for Nonlinear Studies, Los Alamos National Laboratory, Los Alamos, NM 87545, USA*
electronic address: cmb@wustl.edu

[2] *Department of Physics, University of Connecticut, Storrs, CT 06269, USA*
electronic address: philip.mannheim@uconn.edu

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Contrary to common belief, it is shown that theories whose field equations are higher than second order in derivatives need not be stricken with ghosts. In particular, the prototypical fourth-order derivative Pais-Uhlenbeck oscillator model is shown to be free of states of negative energy or negative norm. When correctly formulated (as a $\mathcal{PT}$ symmetric theory), the theory determines its own Hilbert space and associated positive-definite inner product. In this Hilbert space the model is found to be a fully acceptable quantum-mechanical theory that exhibits unitary time evolution.

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It is widely believed that field theories based on equations of motion higher than second order are unacceptable. It is thought that higher-order theories would possess propagators having poles with nonpositive residues. Such poles would be associated with states, known as *ghosts*, which have nonpositive norms and would therefore threaten the unitarity of the theory. The purpose of this paper is to debunk this folklore, and in so doing, to regenerate interest in higher-order quantum field theories. Higher-order field theories are potentially of great value because they can address naturally the renormalization issues connected with elementary particle self-energies and quantum gravitational fluctuations [1].

To understand the issues involved, we review the situation that arises with regard to the Lee model. The Lee model was proposed in 1954 as a trilinearly coupled quantum field theory in which the entire renormalization program can be carried out in closed form [2]. However, just one year later it was argued that this theory has a ghost state [3]. To be precise, a ghost appears to arise in the Lee model when the renormalized coupling constant exceeds a critical value. Above this critical value, the Lee-model Hamiltonian immediately becomes non-Hermitian in the Dirac sense (Dirac Hermitian conjugation means combined matrix transposition and complex conjugation) because its trilinear interaction term acquires an imaginary coefficient. In the non-Hermitian phase of the Lee model a state of negative norm in the Dirac sense emerges.

*Permanent address: Physics Department, Washington University, St. Louis, MO 63130, USA

For the past half century, there have been multiple attempts to make sense of the Lee model as a valid quantum theory (starting as early as [4]), but it was not until 2005 that it was understood that if this theory is formulated correctly, it does not have a ghost at all [5]. The solution to the Lee-model-ghost problem is that when the coupling constant exceeds its critical value, the Hamiltonian undergoes a transition from being Dirac Hermitian to being $\mathcal{PT}$ symmetric, that is, symmetric under combined parity reflection and time reversal. In addition, in the sector of the Lee model in which the ghost appears, the $\mathcal{PT}$ symmetry is not broken in the sense that all energy eigenvalues are real. For any $\mathcal{PT}$-symmetric Hamiltonian having an unbroken $\mathcal{PT}$ symmetry, it is necessary to introduce a completely new Hilbert-space inner product [6, 7, 8, 9, 10]. With respect to the new inner product appropriate for the $\mathcal{PT}$-symmetric phase of the Lee model, the Hamiltonian becomes self-adjoint, and its so-called ghost state becomes an ordinary quantum state with positive $\mathcal{PT}$ norm. This same procedure has been applied to other seemingly problematic models [11].

The purpose of this paper is to show that this same prescription not only can, but in fact must be implemented in higher-derivative field theories as well, and that the negative Dirac-norm states that arise in these theories can instead really be ordinary quantum states having positive $\mathcal{PT}$ norm. Specifically, we will use the same technique developed for the Lee model to study the fourth-order quantum-mechanical Pais-Uhlenbeck oscillator, a model which is the prototypical higher-derivative quantum field theory. We will show by explicit construction that this model is actually a $\mathcal{PT}$ symmetric theory that is totally free of negative-norm states.

The action that defines the Pais-Uhlenbeck model is acceleration-dependent [12]:

$$I_{\rm PU}=\frac{\gamma}{2}\int dt\left[\ddot{z}^2-\left(\omega_1^2+\omega_2^2\right)\dot{z}^2+\omega_1^2\omega_2^2z^2\right], \qquad (1)$$

where γ, ω_1, and ω_2 are all positive constants and without loss of generality we take $\omega_1\geq\omega_2$. This model represents two oscillators coupled by a fourth-order equation of motion of the form $d^4z/dt^4+(\omega_1^2+\omega_2^2)d^2z/dt^2+\omega_1^2\omega_2^2z=0$ [13]. With $\dot{z}$ serving as the canonical conjugate of both z and $\ddot{z}$, the system is constrained and its Hamiltonian must be found by the method of Dirac constraints. To this end, in place of $\dot{z}$ we introduce a new dynamical variable x (with corresponding conjugate p_x), and via the Dirac method construct the Hamiltonian [14, 15, 16]

$$H=\frac{p_x^2}{2\gamma}+p_zx+\frac{\gamma}{2}\left(\omega_1^2+\omega_2^2\right)x^2-\frac{\gamma}{2}\omega_1^2\omega_2^2z^2. \qquad (2)$$

This Hamiltonian depends on two coordinates x and z, and their canonical conjugates, p_x and p_z. Using this Hamiltonian, the Poisson-bracket algebra of the operators x, p_x, z, and p_z is found to be closed with its nonzero elements given by $\{x,p_x\}=1$ and $\{z,p_z\}=1$. This construction makes no reference to the classical equations of motion and thus holds for both stationary and nonstationary classical paths. Consequently, we can use it to quantize the theory, with the nonzero quantum commutators being given by $[x,p_x]=i$ and $[z,p_z]=i$.

To construct a Fock-space representation of the theory, we introduce two sets of creation and annihilation operators according to

$$\begin{aligned} z &= a_1+a_1^{\dagger}+a_2+a_2^{\dagger}, \\ p_z &= i\gamma\omega_1\omega_2^2(a_1-a_1^{\dagger})+i\gamma\omega_1^2\omega_2(a_2-a_2^{\dagger}), \\ x &= -i\omega_1(a_1-a_1^{\dagger})-i\omega_2(a_2-a_2^{\dagger}), \\ p_x &= -\gamma\omega_1^2(a_1+a_1^{\dagger})-\gamma\omega_2^2(a_2+a_2^{\dagger}), \end{aligned} \qquad (3)$$

This yields a Hamiltonian of the form [14, 15, 16]

$$H=2\gamma(\omega_1^2-\omega_2^2)(\omega_1^2a_1^{\dagger}a_1-\omega_2^2a_2^{\dagger}a_2)+\tfrac{1}{2}(\omega_1+\omega_2), \qquad (4)$$

with the nonzero Fock-space commutators being given by

$$\omega_1[a_1,a_1^{\dagger}]=-\omega_2[a_2,a_2^{\dagger}]=\frac{1}{2\gamma\left(\omega_1^2-\omega_2^2\right)}. \qquad (5)$$

The Hamiltonian (4) describes two harmonic oscillators, each with strictly real energy eigenvalues. There are two possible realizations for the theory. If we take a_1 and a_2 to annihilate the no-particle state $|\Omega\rangle$,

$$a_1|\Omega\rangle=0, \qquad a_2|\Omega\rangle=0, \qquad (6)$$

the energy spectrum is then bounded below, and in this case $|\Omega\rangle$ is the ground state with energy $\frac{1}{2}(\omega_1+\omega_2)$. However in this case, the excited state $a_2^{\dagger}|\Omega\rangle$, which lies at energy ω_2 above the ground state, has a Dirac norm $\langle\Omega|a_2a_2^{\dagger}|\Omega\rangle$ which is negative. On the other hand, if a_1 and $a_2^{\dagger}$ annihilate the no-particle state $|\Omega\rangle$,

$$a_1|\Omega\rangle=0, \qquad a_2^{\dagger}|\Omega\rangle=0, \qquad (7)$$

then the theory is free of negative-norm states, but the energy spectrum is unbounded below. Both of these outcomes are unacceptable and characterize the generic problems that are thought to afflict higher-derivative quantum theories.

The above analysis relies on using the standard Dirac norm, but such a norm in quantum theory is actually not mandatory. Specifically, in an eigenvalue equation of the form $H|\psi\rangle=E|\psi\rangle$, the Hamiltonian acts linearly on ket vectors, without any reference to bra vectors at all. Consequently, the eigenvalue equation is unaffected if the bra vector is not taken to be the Dirac conjugate $\langle\psi|$ of the ket $|\psi\rangle$.

To determine what the appropriate bra vector should be, we need to supply some global information. This is best done using the coordinate representation of the canonical commutators, which for the Pais-Uhlenbeck oscillator model are given by

$$p_z=-i\frac{\partial}{\partial z}, \qquad p_x=-i\frac{\partial}{\partial x}. \qquad (8)$$

In this representation the Schrödinger equation $H\psi_n=E_n\psi_n$ takes the form

$$\begin{aligned} \Bigg[-\frac{1}{2\gamma}\frac{\partial^2}{\partial x^2}-ix\frac{\partial}{\partial z}+\frac{\gamma}{2}\left(\omega_1^2+\omega_2^2\right)x^2 \\ -\frac{\gamma}{2}\omega_1^2\omega_2^2z^2\Bigg]\psi_n(z,x)=E_n\psi_n(z,x), \end{aligned} \qquad (9)$$

the state whose energy is $\frac{1}{2}(\omega_1+\omega_2)$ has an eigenfunction of the form [16]

$$\begin{aligned} \psi_0(z,x) &= \exp\Bigg[\frac{\gamma}{2}(\omega_1+\omega_2)\omega_1\omega_2z^2 \\ &\qquad +i\gamma\omega_1\omega_2zx-\frac{\gamma}{2}(\omega_1+\omega_2)x^2\Bigg], \end{aligned} \qquad (10)$$

and the states of higher energy have eigenfunctions that are polynomial functions of x and z times $\psi_0(z,x)$. The eigenfunction $\psi_0(z,x)$ has the defect that it is not normalizable on the real-x and real-z axes; it grows exponentialy as $z\to\pm\infty$. Evidently, the coordinate-space realization of p_z in (8) is not Hermitian.

Since one can only use the realization $p_z=-i\partial/\partial z$ when the $[z,p_z]$ commutator acts on well-behaved test functions, we see that such well-behaved functions cannot be taken to lie on the real z axis. Moreover, the divergence of $\psi_0(z,x)$ in (10) as $|z|\to\infty$ is not restricted to the real axis, and it even occurs in two *Stokes's wedges* in the complex-z plane of angular opening 90° and centered

about the positive- and negative-real axes (the east and west quadrants of the letter X). However, in the complementary 90° Stokes' wedges centered about the positive- and negative-imaginary axes (the north and south quadrants of the letter X), $\psi_0(z,x)$ vanishes exponentially rapidly as $|z|\to\infty$. We must thus restrict the eigenvalue problem (9) to the complementary (north, south) Stokes' wedges, and in so doing we thus take care of the normalization problem for the eigenfunctions. In these wedges $\psi_0(z,x)$ is the fully normalizable ground-state of the system and the energy spectrum is precisely the purely real one associated with the Fock-space option given in (6).

Rather than working on the imaginary axis, it is instructive to perform the 90° rotation $y=-iz$. The modified Hamiltonian with $y=-iz$ (and therefore with $q=ip_z$ to enforce $[y,q]=i$) has the form

$$H=\frac{p^2}{2\gamma}-iqx+\frac{\gamma}{2}\left(\omega_1^2+\omega_2^2\right)x^2+\frac{\gamma}{2}\omega_1^2\omega_2^2y^2, \qquad (11)$$

where for notational simplicity we have replaced p_x by p. In (11) the operators p, x, q, and y are now formally Hermitian [17], but because of the $-iqx$ term, H has become complex and is manifestly not Dirac Hermitian [18]. This non-Hermiticity property is not at all apparent in the original form of the Hamiltonian given in (2). This surprising and completely unexpected emergence of a non-Hermitian term in the Pais-Uhlenbeck Hamiltonian is the root cause of the infamous ghost problem of the Pais-Uhlenbeck model.

While the Hamiltonian in (11) is not Dirac Hermitian, it does fall into a particular class of equally physically viable Hamiltonians, those that are $\mathcal{PT}$ symmetric. To establish this $\mathcal{PT}$ symmetry we make the following assignments: Under $\mathcal{P}$ and $\mathcal{T}$, we take p and x to transform like conventional coordinate and momentum variables. However, we define q and y to transform unconventionally in a way that has not been seen in previous studies of $\mathcal{PT}$-symmetric quantum mechanics; in the language of quantum field theory, q and y transform as parity scalars instead of pseudoscalars, and they have abnormal behavior under time reversal. These transformation properties are summarized in the following table:

	p	x	q	y
$\mathcal{P}$	$-$	$-$	$+$	$+$
$\mathcal{T}$	$-$	$+$	$+$	$-$
$\mathcal{PT}$	$+$	$-$	$+$	$-$

(12)

Because H has an entirely real spectrum, the $\mathcal{PT}$ symmetry of H is unbroken.

Having shown that the $\mathcal{PT}$ symmetry of H is unbroken, we will be able to reinterpret the ghost as a conventional quantum state of positive $\mathcal{PT}$ norm. Specifically, following the standard procedures of $\mathcal{PT}$-symmetric quantum mechanics, we construct the $\mathcal{PT}$ norm by calculating an operator called the $\mathcal{C}$ operator [19]. The $\mathcal{C}$ operator associated with the Hamiltonian of interest in (11) is required to satisfy a characteristic set of three conditions:

$$\mathcal{C}^2=1,\quad [\mathcal{C},\mathcal{PT}]=0,\quad [\mathcal{C},H]=0. \qquad (13)$$

This first two of these conditions are kinematical, while the third is dynamical since it involves the specific Hamiltonian H.

In previous investigations it has been established that $\mathcal{C}$ has the general form $\mathcal{C}=e^{-\mathcal{Q}}\mathcal{P}$, where $\mathcal{Q}$ is a real function of the dynamical variables and is Hermitian in the Dirac sense, and it was found that $\mathcal{Q}$ was odd under a change in sign of the momentum variables and even under a change in sign of the coordinate variables. However, because of the abnormal behaviors of the y and q operators in (12), we find that the exact solution to the three simultaneous algebraic equations in (13) gives an unusual and previously unencountered structure for $\mathcal{Q}$:

$$\mathcal{Q}=\alpha pq+\beta xy, \qquad (14)$$

where α and β are given by

$$\beta\;=\;\gamma^2\omega_1^2\omega_2^2\alpha,\quad \sinh(\sqrt{\alpha\beta})=\frac{2\omega_1\omega_2}{(\omega_1^2-\omega_2^2)}. \qquad (15)$$

Even though the form of $\mathcal{Q}$ in (14) is unprecedented in $\mathcal{PT}$-symmetric quantum mechanics, the effect of performing a similarity transformation on the dynamical variables x,p,y,q using $e^{-\mathcal{Q}}$ still generates a canonical transformation that preserves the commutation relations, as has always been the case in the past. For the Pais-Uhlenbeck Hamiltonian, the transformation is [20]:

$$\begin{aligned}
e^{-\mathcal{Q}}xe^{\mathcal{Q}} &= x\cosh(\sqrt{\alpha\beta})+i\sqrt{\frac{\alpha}{\beta}}\,q\sinh(\sqrt{\alpha\beta}),\\
e^{-\mathcal{Q}}qe^{\mathcal{Q}} &= q\cosh(\sqrt{\alpha\beta})-i\sqrt{\frac{\beta}{\alpha}}\,x\sinh(\sqrt{\alpha\beta}),\\
e^{-\mathcal{Q}}ye^{\mathcal{Q}} &= y\cosh(\sqrt{\alpha\beta})+i\sqrt{\frac{\alpha}{\beta}}\,p\sinh(\sqrt{\alpha\beta}),\\
e^{-\mathcal{Q}}pe^{\mathcal{Q}} &= p\cosh(\sqrt{\alpha\beta})-i\sqrt{\frac{\beta}{\alpha}}\,y\sinh(\sqrt{\alpha\beta}).
\end{aligned} \qquad (16)$$

In $\mathcal{PT}$-symmetric quantum mechanics, performing a similarity transformation on the $\mathcal{PT}$-symmetric Hamiltonian with $e^{-\mathcal{Q}/2}$ yields a positive definite Hamiltonian which is Hermitian in the Dirac sense [21]. In the current context we obtain

$$\begin{aligned}
\tilde{H} &= e^{-\mathcal{Q}/2}He^{\mathcal{Q}/2}\\
&= \frac{p^2}{2\gamma}+\frac{q^2}{2\gamma\omega_1^2}+\frac{\gamma}{2}\omega_1^2x^2+\frac{\gamma}{2}\omega_1^2\omega_2^2y^2.
\end{aligned} \qquad (17)$$

The spectrum of this Hamiltonian is manifestly real and positive. However, because this Hamiltonian is related

to the original Pais-Uhlenbeck Hamiltonian by a similarity transformation, which is isospectral, despite the $-iqx$ term, the positivity of the Pais-Uhlenbeck Hamiltonian is proved.

Furthermore, the eigenstates of $|\tilde{\psi}\rangle$ of $\tilde{H}$ have positive inner product and can be normalized in the conventional Dirac way using the standard inner product:

$$\langle\tilde{\psi}|\tilde{\psi}\rangle = 1, \tag{18}$$

where the bra vector is the Dirac-Hermitian adjoint of the ket vector. Equivalently, for the eigenstates $|\psi\rangle$ of the Hamiltonian H, because the vectors are mapped by

$$|\tilde{\psi}\rangle = e^{-\mathcal{Q}/2}|\psi\rangle, \tag{19}$$

the normalization of the eigenstates of H is given as

$$\langle\psi|e^{-\mathcal{Q}}|\psi\rangle = 1. \tag{20}$$

Thus, it is the norm of (20) that is relevant for the Pais-Uhlenbeck model, with $\langle\psi|e^{-\mathcal{Q}}$ rather than $\langle\psi|$ being the appropriate conjugate for $|\psi\rangle$. Finally, because the norm in (20) is positive and because $[H,\mathcal{CPT}]=0$, the Hamiltonian H generates unitary time evolution.

To conclude, we see that in order to construct the correct Hilbert space for the Pais-Uhlenbeck theory, one must determine the region in the complex plane where operators such as $-i\partial/\partial_z$ act as well-defined Hermitian operators. The appearance of a ghost state when one takes the derivative operator to be Hermitian on the real z axis is not an indication that there is anything wrong with the theory itself, but only with the way it is being analysed. Consequently, if treated properly, higher derivative theories such as conformal gravity have the potential to be completely viable as quantum theories of gravity in four spacetime dimensions [22].

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[17] Despite the introduction of $y=-iz$ and $q=ip_z$, since y and q are Hermitian operators, their c-number eigenvalue coordinates and momenta are strictly real.

[18] Although quantum mechanics is usually formulated in terms of Hermitian operators, Hermiticity is only sufficient to give real eigenvalues but not necessary. Specifically, while a Hermitian operator has real eigenvalues, there is no converse theorem that says that the eigenvalues of a non-Hermitian operator are not real. The Hamiltonian (11) is an example of a non-Hermitian operator that possesses a strictly real spectrum of eigenvalues.

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